Name:

SA402 · Dynamic and Stochastic Models | Fall 2022 · Uhan

Exam 2 – 11/9/2022

Instructions

- You have 50 minutes to complete this exam.
- You may use your plebe-issue TI-36X Pro calculator.
- You may not use any other materials.
- **No collaboration allowed.** All work must be your own.
- **Show all your work.** To receive full credit, your solutions must be completely correct, sufficiently justified, and easy to follow.
- Keep this booklet intact.
- **Do not discuss the contents of this exam with any midshipmen until it is returned to you.**

Problem	Weight	Score
1a	1.25	
1b	1.25	
1c	1.25	
2	1.25	
3a	1.25	
3b	1.25	
3c	1.25	
4	1.25	
Total		/ 100

Problem 0. Copy and sign the honor statement below. This exam will not be graded without a signed honor statement.

> The Naval Service I am a part of is bound by honor and integrity. I will not compromise our values by giving or receiving unauthorized help on this exam.

Problem 1. Lemon is a scooter sharing service that operates in Simplexville. They have hired you to study the movement of its scooters between three regions in Simplexville: Downtown, North, and South. The company's data science team has modeled the movement of a scooter as a Markov chain with 3 states. The states $1, 2, 3$ correspond to Downtown, North, and South, respectively, and each time step corresponds to one scooter trip. When a scooter reaches its destination, it stays in the destination region until it is used again. The one-step transition probability matrix for a scooter is:

$$\mathbf{P} = \begin{bmatrix} 0.65 & 0.25 & 0.10 \\ 0.30 & 0.50 & 0.20 \\ 0.35 & 0.20 & 0.45 \end{bmatrix}$$

a. Suppose at the beginning of the day, 50% of the bikes are in the Downtown region, 25% in the North region, and 25% in the South region. What is the probability that a bike randomly chosen at the beginning of the day will be in the Downtown region after 5 trips?

b. What is the probability that a scooter starting in the South region will be in the Downtown region after 3 trips?

Name:

Here is the transition probability matrix for Problem 1 again:

$$\mathbf{P} = \begin{bmatrix} 0.65 & 0.25 & 0.10 \\ 0.30 & 0.50 & 0.20 \\ 0.35 & 0.20 & 0.45 \end{bmatrix}$$

c. What is the probability that a bike starts in the Downtown region, stays in either the Downtown or South regions for 6 trips, and then goes to the North region in the 7th trip?

Problem 2. You have just been hired as an analyst in the Cauchy County Department of Health and Human Services. Your predecessor developed a model of the county population, in which each citizen can be classified as living in one of three location types: urban, rural, or suburban. In their model, the state of the system is defined as a citizen's current location type, and the time index is defined to be 1 year.

Describe what assumptions need to be made in order for the time-stationarity property to hold. (You do not need to discuss whether these assumptions are realistic.)

Problem 3. An autonomous UUV has been programmed to move randomly between 6 regions according to a Markov chain. Looking at the documentation written by the programmer, you find the following one-step transition matrix:

$$\mathbf{P} = \begin{bmatrix} 0.10 & 0.25 & 0.15 & 0.10 & 0.05 & 0.35 \\ 0.20 & 0.10 & 0.40 & 0.10 & 0.15 & 0.05 \\ 0 & 0 & 0.25 & 0.75 & 0 & 0 \\ 0 & 0 & 0.65 & 0.35 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.70 & 0.30 \\ 0 & 0 & 0 & 0 & 0.50 & 0.50 \end{bmatrix}$$

a. Do regions 1 and 2 form an irreducible set of states? Why or why not?

b. Suppose the UUV reaches region 4. What is the long-run fraction of time it spends in region 4?

Name:

Here is the transition probability matrix for Problem 3 again:

$$\mathbf{P} = \begin{bmatrix} 0.10 & 0.25 & 0.15 & 0.10 & 0.05 & 0.35 \\ 0.20 & 0.10 & 0.40 & 0.10 & 0.15 & 0.05 \\ 0 & 0 & 0.25 & 0.75 & 0 & 0 \\ 0 & 0 & 0.65 & 0.35 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.70 & 0.30 \\ 0 & 0 & 0 & 0 & 0.50 & 0.50 \end{bmatrix}$$

c. Suppose the UUV starts in region 2. What is the probability that the UUV eventually ends up in region 5 or 6?

Problem 4. You have been put in charge of inventory management at the Poisson Fish Market. The inventory system for tuna works as follows:

1. Observe the number of crates of tuna in inventory at the beginning of the day. Call this number n. The storage area for tuna can hold at most 4 crates.
2. If $n \in \{0, 1\}$, then order $4 - n$ crates. If $n \in \{2, 3, 4\}$, then order 0 crates. Orders are delivered immediately.
3. Throughout the day, some of these crates of tuna are sold. With probability 1/3, 0 crates are sold. With probability 1/2, 1 crate is sold. With probability 1/6, 2 crates are sold.
4. Observe the number of crates in inventory at the beginning of the next day.

When you start observing the system, there are 2 crates of tuna in inventory.

Model this setting as a Markov chain by defining:

- the state space,
- the meaning of one time step in the setting's context,
- the meaning of the state visited in the nth time step in the setting's context, and
- the one-step transition probabilities and initial state probabilities.